example

find the points on the sphere $x^2 + y^2 + z^2 = 4$ closest to
& farthest from $(3, -1, 1)$

optimize distance, subject to sphere

we'll optimize distance squared

$$\begin{cases} \text{optimize: } f(x, y, z) = (x-3)^2 + (y+1)^2 + (z-1)^2. \\ \text{subject to: } x^2 + y^2 + z^2 = 4 \end{cases}$$

$$\begin{cases} \text{OPT: } (x^2 - 6x + 9) + (y^2 + 2y + 1) + (z^2 - 2z + 1) \\ \text{SUB to: } x^2 + y^2 + z^2 = 4 \end{cases}$$

$$\begin{cases} \text{OPT: } (x^2 + y^2 + z^2) + (9 + 1 + 1) + (-6x + 2y - 2z) \\ \text{SUB: } x^2 + y^2 + z^2 - 4 = 0 \end{cases}$$

$$\begin{cases} \text{OPT: } f(x, y, z) = 15 - 6x + 2y - 2z \\ \text{SUB: } g(x, y, z) = 0 \quad g(x, y, z) = x^2 + y^2 + z^2 - 4 \end{cases}$$

$F(x, y, z, \lambda) = f(x, y, z) - \lambda g(x, y, z) = 15 - 6x + 2y - 2z - \lambda(x^2 + y^2 + z^2 - 4)$

$\nabla F = \langle -6 - 2\lambda x, \ 2 - 2\lambda y, \ -2 - 2\lambda z, \ -(x^2 + y^2 + z^2 - 4) \rangle$

$\nabla F = \vec{0}$ iff
$$\begin{cases} -6 - 2\lambda x = 0 \\ 2 - 2\lambda y = 0 \\ -2 - 2\lambda z = 0 \\ -(x^2 + y^2 + z^2 - 4) = 0 \end{cases}$$
iff
$$\begin{cases} \lambda x = -3 & (1) \\ \lambda y = 1 & (2) \\ \lambda z = -1 & (3) \\ x^2 + y^2 + z^2 = 4 & (4) \end{cases}$$

$\lambda \neq 0$ by (1)

multiply (4) by λ^2
$\lambda^2(x^2 + y^2 + z^2) = 4\lambda^2 \ \longrightarrow \ (\lambda x)^2 + (\lambda y)^2 + (\lambda z)^2 = 4\lambda^2$

$$(\lambda x)^2 + (\lambda y)^2 + (\lambda z)^2 = 4\lambda^2$$
$$(-3)^2 + (1)^2 + (-1)^2 = 4\lambda^2$$
$$\lambda = \pm\sqrt{11}/2$$

2 cases:

if $\lambda = \sqrt{11}/2$ solving (1,2,3) for x,y,z

obtain point $A = (-6/\sqrt{11}, \ 2/\sqrt{11}, \ -2/\sqrt{11})$

$f(A) = 15 - 6(-6/\sqrt{11}) + 2(2/\sqrt{11}) - 2(-2/\sqrt{11})$
$= 15 + 36/\sqrt{11} + 4/\sqrt{11} + 4/\sqrt{11}$
$= 15 + 44/\sqrt{11}$

if $\lambda = -\sqrt{11}/2$ solving (1,2,3) for x,y,z

obtain point $B = (6/\sqrt{11}, \ -2/\sqrt{11}, \ 2/\sqrt{11})$

$f(B) = 15 - 6(6/\sqrt{11}) + 2(-2\sqrt{11}) - 2(2/\sqrt{11})$
$= 15 - 44/\sqrt{11}$

$f(A)$ is max. distance^2 & $f(B)$ is min. distance^2
b/c $f(A) \perp f(B)$ by Lagrange Multipliers

A is furthest
B is closest

exercise
a box is to be built with surface area 12 what is
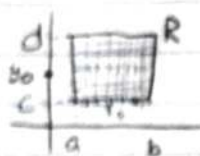max. volume of rectangular box without a lid?

Chaptr 15
15.1 Double Integrals

idea: we have functions of several variables. How do we integrate them?

in calc III: a definite integral over region R should be the "net volume of graph of f over R"

$$\iint_R f\, dA: \text{volume of}$$

$R = [a,b] \times [c,d]$
$= \{(x,y): x \in (a,b), y \in (c,d)\}$

fixing y_0 means working with single variable function $f(x, y_0)$ (and so the same fixing x)

can approximate volume of R by using "left-lower corner points" to determine height of boxes via f (corner point). limiting these approximations yields the true volume....

Fubini's Theorem:
if $f(x,y)$ is continuous on rectangle $R = [a,b] \times [c,d]$, then

$$\int_{y=c}^{d}\left(\int_{x=a}^{b} f(x,y)\, dx\right) dy = \iint_R f(x,y)\, dA = \int_{x=a}^{b}\left(\int_{y=c}^{d} f(x,y)\, dy\right) dx$$

example

compute $\iint_R x\sec^2(y)\, dA$ for $R = [0,2] \times [0, \pi/4]$

$$\iint_R x\sec^2(y)\, dA = \int_{y=0}^{\pi/4} \int_{x=0}^{2} x\sec^2(y)\, dx\, dy$$

$$\int_{x=0}^{2} x\sec^2(y)\, dx = \sec^2(y)\int_0^2 x\, dx = \sec^2(y)\left(\frac{x^2}{2}\right)\Big|_0^2$$

$$= \sec^2(y)\frac{1}{2}(4-0) = 2\sec^2(y)$$

$$\int_{y=0}^{\pi/4} 2\sec^2(y)\, dy = 2\tan(y)\Big|_0^{\pi/4} = 2(\tan\tfrac{\pi}{4} - \tan 0)$$

$$= 2(1-0) = 2$$

Or

$$\iint_R x\sec^2(y)\, dA = \int_{x=0}^{2} \int_{y=0}^{\pi/4} x\sec^2(y)\, dy\, dx$$

$$\int_{y=0}^{\pi/4} x\sec^2(y)\, dy = x(\tan(y))\Big|_0^{\pi/4} = x(\tan\tfrac{\pi}{4} - \tan 0)$$

$$= x(1-0) = x$$

$$\int_{x=0}^{2} x\, dx = \frac{x^2}{2}\Big|_0^2 = \frac{4}{2} - \frac{0}{2} = 2$$

example

compute $\iint_R \frac{1}{1+x+y}$ for $R = [1,2] \times [2,3]$

$$\iint_R \frac{1}{1+x+y} \, dA = \int_{x=1}^{2} \int_{y=2}^{3} \frac{1}{1+x+y} \, dy \, dx$$

$$\int_{y=2}^{3} \frac{1}{1+x+y} \, dy \qquad \begin{array}{l} u = 1+x+y \\ du = dy \end{array}$$

$$\int_{y=2}^{3} \frac{1}{u} \, du = \ln|u| \Big|_{2}^{3} = \ln|1+x+y| \Big|_{2}^{3}$$

$$= \ln|4+x| - \ln|3+x|$$

$$\int_{x=1}^{2} (\ln|4+x| - \ln|3+x|) \, dx$$

$$= (4+x)(\ln(4+x) - 1) - (3+x)(\ln(3+x) - 1) \Big|_{1}^{2}$$